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FRACTIONAL HERMITE HADAMARD'S TYPE INEQUALITY FOR THE CO-ORDINATED CONVEX FUNCTIONS

TUBA TUNÇ[1], MEHMET ZEKİ SARIKAYA[1], HATİCE YALDIZ[2]

Abstract. In this paper, we consider the co-ordinated convex functions and obtain some Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals. For this purpose, we first prove an supplement al result for two variables. Using this auxiliary result, integral inequalities for the left-hand side of the fractional Hermite-Hadamard type inequality on the coordinates are derived. These represent can be viewed as a refinement of the previously known results.

Keywords: Riemann-Liouville fractional integrals, convex function, co-ordinated convex mapping, Hermite-Hadamard inequality, Hölder's inequality.

AMS Subject Classification: 26A33, 26A51, 26D15.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality holds:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w),$$

for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$. A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [10]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.1. *A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated convex on Δ, for all $t, s \in [0,1]$ and $(x,y), (u,w) \in \Delta$, if the following inequality holds:*

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ \leq\ & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,w) + (1-t)(1-s)f(y,w). \end{aligned} \tag{1}$$

[1]Department of Mathematics, Faculty of Science and Arts, Düzce University, Turkey
[2]Department of Mathematics, Faculty of Science, Karamanoglu Mehmetbey University, Karaman, Turkey
e-mail: tubatunc03@gmail.com, sarikayamz@gmail.com, yaldizhatice@gmail.com
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Clearly, every convex function is co-ordinated convex. Furthermore, there exist a co-ordinated convex function, which is not convex (see, [10]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1, 2, 3, 5, 10, 13-15, 17, 18, 23, 25]). In [23], Sarikaya and Yaldiz proved inequalities of the Hermite-Hadamard type by using the definition of co-ordinated convex functions for L-Lipschitzian mappings. In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory, which are used further in this paper. More details, one can consult [11, 12, 16].

Definition 1.2. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Definition 1.3. *Let $f \in L_1([a,b] \times [c,d])$. The Riemann-Liouville integrals $J_{a+,c+}^{\alpha,\beta}$, $J_{a+,d-}^{\alpha,\beta}$, $J_{b-,c+}^{\alpha,\beta}$ and$J_{b-,d-}^{\alpha,\beta}$ of order $\alpha, \beta > 0$ with $a, c \geq 0$ are defined by*

$$J_{a+,c+}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y (x-t)^{\alpha-1} (y-s)^{\beta-1} f(t,s)dsdt, \quad x > a, \ y > c,$$

$$J_{a+,d-}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_y^d (x-t)^{\alpha-1} (s-y)^{\beta-1} f(t,s)dsdt, \quad x > a, \ y < d,$$

$$J_{b-,c+}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_c^y (t-x)^{\alpha-1} (y-s)^{\beta-1} f(t,s)dsdt, \quad x < b, \ y > c,$$

and

$$J_{b-,d-}^{\alpha,\beta} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_x^b \int_y^d (t-x)^{\alpha-1} (s-y)^{\beta-1} f(t,s)dsdt, \quad x < b, \ y < d,$$

respectively. Here, Γ is the Gamma function,

$$J_{a+,c+}^{0,0} f(x,y) = J_{a+,d-}^{0,0} f(x,y) = J_{b-,c+}^{0,0} f(x,y) = J_{b-,d-}^{0,0} f(x,y) = f(x,y),$$

and

$$J_{a+,c+}^{1,1} f(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^x \int_c^y f(t,s)dsdt.$$

Similar to Definition 1.2 and Definition 1.3, we introduce the following fractional integrals:

$$J_{a+}^{\alpha} f\left(x, \frac{c+d}{2}\right) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f\left(t, \frac{c+d}{2}\right) dt, \quad x > a,$$

$$J_{b-}^{\alpha} f\left(x, \frac{c+d}{2}\right) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1} f\left(t, \frac{c+d}{2}\right) dt, \quad x < b,$$

$$J_{c+}^{\beta} f\left(\frac{a+b}{2}, y\right) = \frac{1}{\Gamma(\beta)} \int_{c}^{y} (y-s)^{\beta-1} f\left(\frac{a+b}{2}, s\right) ds, \quad y > c,$$

$$J_{d-}^{\beta} f\left(\frac{a+b}{2}, y\right) = \frac{1}{\Gamma(\beta)} \int_{y}^{d} (s-y)^{\beta-1} f\left(\frac{a+b}{2}, s\right) ds, \quad y < d.$$

Remarkable, Sarikaya et al.([22]) and ([24]) gave the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals by using convex functions of two variables on the co-ordinates.

Theorem 1.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ is co-ordinated convex on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \leq a < b,\ 0 \leq c < d$ and $f \in L_1(\Delta)$. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \qquad (2)\\
\leq\ & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f\left(b, \frac{c+d}{2}\right) + J_{b-}^{\alpha} f\left(a, \frac{c+d}{2}\right) \right] \\
& + \frac{\Gamma(\beta+1)}{2(d-c)^{\beta}} \left[J_{c+}^{\beta} f\left(\frac{a+b}{2}, d\right) + J_{d-}^{\beta} f\left(\frac{a+b}{2}, c\right) \right] \\
\leq\ & \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}} \left[J_{a+,c+}^{\alpha,\beta} f(b,d) + J_{a+,d-}^{\alpha,\beta} f(b,c) + J_{b-,c+}^{\alpha,\beta} f(a,d) + J_{b-,d-}^{\alpha,\beta} f(a,c) \right] \\
\leq\ & \frac{\Gamma(\alpha+1)}{4(b-a)^{\alpha}} \left[J_{a+}^{\alpha} f(b,c) + J_{a+}^{\alpha} f(b,d) + J_{b-}^{\alpha} f(a,c) + J_{b-}^{\alpha} f(a,d) \right] \\
& + \frac{\Gamma(\beta+1)}{4(d-c)^{\beta}} \left[J_{c+}^{\beta} f(a,d) + J_{c+}^{\beta} f(b,d) + J_{d-}^{\beta} f(a,c) + J_{d-}^{\beta} f(b,c) \right] \\
\leq\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4},
\end{aligned}$$

where Γ is the Gamma function.

For some recent results connected with fractional integral inequalities, see ([4-9, 19-22]).

The paper aims to establish new Hermite-Hadamard type inequalities for co-ordinated convex on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ via Riemann-Liouville fractional integrals. Firstly, we will give an identity for two variables, and with the help of this fractional type integral identity, we will provide some integral inequalities connected with the left-hand side of the Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals.

2. Fractional inequalities for co-ordinated convex functions

To make the presentation more comfortable and compact to understand; we make some symbolic representation:

$$
\begin{aligned}
I_1 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sc+(1-s)d\right) dsdt, \\
I_2 &= -\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^\alpha \left(1-s^\beta\right) \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sc+(1-s)d\right) dsdt, \\
I_3 &= -\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^\alpha) s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sc+(1-s)d\right) dsdt, \\
I_4 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^\alpha)\left(1-s^\beta\right) \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sc+(1-s)d\right) dsdt, \\
I_5 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)a, sd+(1-s)c\right) dsdt, \\
I_6 &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^\alpha \left(s^\beta-1\right) \frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)a, sd+(1-s)c\right) dsdt, \\
I_7 &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (t^\alpha-1) s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)a, sd+(1-s)c\right) dsdt, \\
I_8 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (t^\alpha-1)\left(s^\beta-1\right) \frac{\partial^2 f}{\partial t \partial s}\left(tb+(1-t)a, sd+(1-s)c\right) dsdt, \\
I_9 &= -\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sd+(1-s)c\right) dsdt, \\
I_{10} &= -\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^\alpha \left(s^\beta-1\right) \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sd+(1-s)c\right) dsdt, \\
I_{11} &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^\alpha) s^\beta \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sd+(1-s)c\right) dsdt, \\
I_{12} &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^\alpha)\left(s^\beta-1\right) \frac{\partial^2 f}{\partial t \partial s}\left(ta+(1-t)b, sd+(1-s)c\right) dsdt,
\end{aligned}
$$

$$
\begin{aligned}
I_{13} &= -\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\,dsdt, \\
I_{14} &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\,dsdt, \\
I_{15} &= -\int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (t^{\alpha}-1)s^{\beta}\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\,dsdt, \\
I_{16} &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (t^{\alpha}-1)\left(1-s^{\beta}\right)\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\,dsdt, \\
L_1 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\,dsdt\right|, \\
L_2 &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\,dsdt\right|, \\
L_3 &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^{\alpha})s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\,dsdt\right|, \\
L_4 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\,dsdt\right|, \\
L_5 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sd+(1-s)c)\,dsdt\right|, \\
L_6 &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sd+(1-s)c)\,dsdt\right|, \\
L_7 &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^{\alpha})s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sd+(1-s)c)\,dsdt\right|, \\
L_8 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sd+(1-s)c)\,dsdt\right|,
\end{aligned}
$$

$$L_9 = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^\alpha s^\beta \left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sd+(1-s)c)\, dsdt\right|,$$

$$L_{10} = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^\alpha \left(1-s^\beta\right) \left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sd+(1-s)c)\, dsdt\right|,$$

$$L_{11} = \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^\alpha) s^\beta \left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sd+(1-s)c)\, dsdt\right|,$$

$$L_{12} = \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^\alpha) \left(1-s^\beta\right) \left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sd+(1-s)c)\, dsdt\right|,$$

$$L_{13} = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^\alpha s^\beta \left|\frac{\partial^2 f}{\partial t \partial s}(tb+(1-t)a, sc+(1-s)d)\, dsdt\right|,$$

$$L_{14} = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^\alpha \left(1-s^\beta\right) \left|\frac{\partial^2 f}{\partial t \partial s}(tb+(1-t)a, sc+(1-s)d)\, dsdt\right|,$$

$$L_{15} = \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^\alpha) s^\beta \left|\frac{\partial^2 f}{\partial t \partial s}(tb+(1-t)a, sc+(1-s)d)\, dsdt\right|,$$

$$L_{16} = \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^\alpha) \left(1-s^\beta\right) \left|\frac{\partial^2 f}{\partial t \partial s}(tb+(1-t)a, sc+(1-s)d)\, dsdt\right|.$$

In order to prove our main results, we need the following lemma.

Lemma 2.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $0 \le a < b,\ 0 \le c < d$. If $\frac{\partial^2 f}{\partial t \partial s} \in L(\Delta)$, then the following equality holds:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \left[\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+} f\left(b, \frac{c+d}{2}\right) + J^\alpha_{b-} f\left(a, \frac{c+d}{2}\right)\right]\right. \quad (3)$$

$$\left. + \frac{\Gamma(\beta+1)}{2(d-c)^\beta}\left[J^\beta_{c+} f\left(\frac{a+b}{2}, d\right) + J^\beta_{d-} f\left(\frac{a+b}{2}, c\right)\right]\right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^\alpha (d-c)^\beta}$$

$$\times \left[J^{\alpha,\beta}_{a+,c+} f(b,d) + J^{\alpha,\beta}_{a+,d-} f(b,c) + J^{\alpha,\beta}_{b-,c+} f(a,d) + J^{\alpha,\beta}_{b-,d-} f(a,c)\right] = \frac{(b-a)(d-c)}{4}\sum_{k=1}^{16} I_k.$$

Proof. By integration by parts, we get

$$I_1 = \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^\alpha s^\beta \frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)\, dsdt \quad (4)$$

$$= \int_0^{\frac{1}{2}} s^\beta \left\{ t^\alpha \ \frac{1}{a-b} \frac{\partial f}{\partial s} \left(ta + (1-t)b, sc + (1-s)d \right) \Big|_0^{\frac{1}{2}} \right.$$

$$\left. - \frac{\alpha}{a-b} \int_0^{\frac{1}{2}} t^{\alpha-1} \frac{\partial f}{\partial s} \left(ta + (1-t)b, sc + (1-s)d \right) dt \right\} ds$$

$$= \int_0^{\frac{1}{2}} s^\beta \left\{ -\frac{1}{2^\alpha (b-a)} \frac{\partial f}{\partial s} \left(\frac{a+b}{2}, sc + (1-s)d \right) \right.$$

$$\left. + \frac{\alpha}{b-a} \int_0^{\frac{1}{2}} t^{\alpha-1} \frac{\partial f}{\partial s} \left(ta + (1-t)b, sc + (1-s)d \right) dt \right\} ds$$

$$= -\frac{1}{b-a} \int_0^{\frac{1}{2}} s^\beta \frac{\partial f}{\partial s} \left(\frac{a+b}{2}, sc + (1-s)d \right) ds$$

$$+ \frac{\alpha}{b-a} \int_0^{\frac{1}{2}} t^{\alpha-1} \left[\int_0^{\frac{1}{2}} s^\beta \frac{\partial f}{\partial s} \left(ta + (1-t)b, sc + (1-s)d \right) ds \right] dt$$

$$- \frac{1}{2^{\alpha+\beta} (b-a)(d-c)} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right)$$

$$- \frac{\beta}{2^\alpha (b-a)(d-c)} \int_0^{\frac{1}{2}} s^{\beta-1} f \left(\frac{a+b}{2}, sc + (1-s)d \right) ds$$

$$- \frac{\alpha}{2^\beta (b-a)(d-c)} \int_0^{\frac{1}{2}} t^{\alpha-1} f \left(ta + (1-t)b, \frac{c+d}{2} \right) dt$$

$$+ \frac{\alpha\beta}{(b-a)(d-c)}$$

$$\times \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^{\alpha-1} s^{\beta-1} f \left(ta + (1-t)b, sc + (1-s)d \right) ds dt.$$

Thus, similarly, by integration by parts it follows that

$$I_2 = -\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 t^\alpha \left(1 - s^\beta \right) \frac{\partial^2 f}{\partial t \partial s} \left(ta + (1-t)b, sc + (1-s)d \right) ds dt \tag{5}$$

$$= \frac{\left(2^\beta - 1 \right)}{2^{\alpha+\beta} (b-a)(d-c)} f \left(\frac{a+b}{2}, \frac{c+d}{2} \right)$$

$$- \frac{\beta}{2^\alpha (b-a)(d-c)} \int_{\frac{1}{2}}^1 s^{\beta-1} f \left(\frac{a+b}{2}, sc + (1-s)d \right) ds$$

$$
\begin{aligned}
&-\frac{\alpha\left(2^{\beta}-1\right)}{2^{\beta}\left(b-a\right)\left(d-c\right)}\int\limits_{0}^{\frac{1}{2}} t^{\alpha-1}\left(ta+(1-t)b,\frac{c+d}{2}\right)dt\\
&+\frac{\alpha\beta}{\left(b-a\right)\left(d-c\right)}\int\limits_{0}^{\frac{1}{2}}\int\limits_{\frac{1}{2}}^{1} t^{\alpha-1}s^{\beta-1}f\left(ta+(1-t)b,sc+(1-s)d\right)dsdt,
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= -\int\limits_{\frac{1}{2}}^{1}\int\limits_{0}^{\frac{1}{2}}(1-t^{\alpha})s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b,sc+(1-s)d\right)dsdt \qquad (6)\\
&= \frac{\left(2^{\alpha}-1\right)}{2^{\alpha+\beta}\left(b-a\right)\left(d-c\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{\beta\left(2^{\alpha}-1\right)}{2^{\alpha}\left(b-a\right)\left(d-c\right)}\int\limits_{0}^{\frac{1}{2}} s^{\beta-1}f\left(\frac{a+b}{2},sc+(1-s)d\right)ds\\
&-\frac{\alpha}{2^{\beta}\left(b-a\right)\left(d-c\right)}\int\limits_{\frac{1}{2}}^{1} t^{\alpha-1}f\left(ta+(1-t)b,\frac{c+d}{2}\right)dt\\
&+\frac{\alpha\beta}{\left(b-a\right)\left(d-c\right)}\int\limits_{\frac{1}{2}}^{1}\int\limits_{0}^{\frac{1}{2}} t^{\alpha-1}s^{\beta-1}f\left(ta+(1-t)b,sc+(1-s)d\right)dsdt,
\end{aligned}
$$

$$
\begin{aligned}
I_4 &= \int\limits_{\frac{1}{2}}^{1}\int\limits_{\frac{1}{2}}^{1}(1-t^{\alpha})\left(1-s^{\beta}\right)\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b,sc+(1-s)d\right)dsdt \qquad (7)\\
&-\frac{\left(2^{\alpha}-1\right)\left(2^{\beta}-1\right)}{2^{\alpha+\beta}\left(b-a\right)\left(d-c\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{\beta\left(2^{\alpha}-1\right)}{2^{\alpha}\left(b-a\right)\left(d-c\right)}\int\limits_{\frac{1}{2}}^{1} s^{\beta-1}f\left(\frac{a+b}{2},sc+(1-s)d\right)ds\\
&-\frac{\alpha\left(2^{\beta}-1\right)}{2^{\beta}\left(b-a\right)\left(d-c\right)}\int\limits_{\frac{1}{2}}^{1} t^{\alpha-1}f\left(ta+(1-t)b,\frac{c+d}{2}\right)dt\\
&+\frac{\alpha\beta}{\left(b-a\right)\left(d-c\right)}\int\limits_{\frac{1}{2}}^{1}\int\limits_{\frac{1}{2}}^{1} t^{\alpha-1}s^{\beta-1}f\left(ta+(1-t)b,sc+(1-s)d\right)dsdt,
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= \int\limits_{0}^{\frac{1}{2}}\int\limits_{0}^{\frac{1}{2}} t^{\alpha}s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a,sd+(1-s)c\right)dsdt \qquad (8)\\
&= \frac{1}{2^{\alpha+\beta}\left(b-a\right)\left(d-c\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)
\end{aligned}
$$

$$-\frac{\beta}{2^{\alpha}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} s^{\beta-1} f\left(\frac{a+b}{2}, sd+(1-s)c\right) ds$$

$$-\frac{\alpha}{2^{\beta}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} t^{\alpha-1} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt$$

$$+\frac{\alpha\beta}{(b-a)(d-c)}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} t^{\alpha-1}s^{\beta-1} f\left(tb+(1-t)a, sd+(1-s)c\right) dsdt,$$

$$\begin{aligned} I_6 &= \int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sd+(1-s)c\right) dsdt \qquad (9)\\ &= \frac{\left(2^{\beta}-1\right)}{2^{\alpha+\beta}(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\\ &- \frac{\beta}{2^{\alpha}(b-a)(d-c)}\int_{\frac{1}{2}}^{1} s^{\beta-1} f\left(\frac{a+b}{2}, sd+(1-s)c\right) ds \end{aligned}$$

$$-\frac{\alpha\left(2^{\beta}-1\right)}{2^{\beta}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} t^{\alpha-1} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt$$

$$+\frac{\alpha\beta}{(b-a)(d-c)}\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha-1}s^{\beta-1} f\left(tb+(1-t)a, sd+(1-s)c\right) dsdt,$$

$$I_7 = \int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} (t^{\alpha}-1)s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sd+(1-s)c\right) dsdt \qquad (10)$$

$$= \frac{(2^{\alpha}-1)}{2^{\alpha+\beta}(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$-\frac{\beta(2^{\alpha}-1)}{2^{\alpha}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} s^{\beta-1} f\left(\frac{a+b}{2}, sd+(1-s)c\right) ds$$

$$-\frac{\alpha}{2^{\beta}(b-a)(d-c)}\int_{\frac{1}{2}}^{1} t^{\alpha-1} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt$$

$$+\frac{\alpha\beta}{(b-a)(d-c)}\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} t^{\alpha-1}s^{\beta-1} f\left(tb+(1-t)a, sd+(1-s)c\right) dsdt,$$

$$\begin{aligned}
I_8 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}\left(t^{\alpha}-1\right)\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sd+(1-s)c\right)dsdt \qquad (11)\\
&= \frac{\left(2^{\alpha}-1\right)\left(2^{\beta}-1\right)}{2^{\alpha+\beta}\left(b-a\right)\left(d-c\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&- \frac{\beta\left(2^{\alpha}-1\right)}{2^{\alpha}\left(b-a\right)\left(d-c\right)}\int_{\frac{1}{2}}^{1}s^{\beta-1}f\left(\frac{a+b}{2}, sd+(1-s)c\right)ds\\
&- \frac{\alpha\left(2^{\beta}-1\right)}{2^{\beta}\left(b-a\right)\left(d-c\right)}\int_{\frac{1}{2}}^{1}t^{\alpha-1}f\left(tb+(1-t)a, \frac{c+d}{2}\right)dt\\
&+ \frac{\alpha\beta}{\left(b-a\right)\left(d-c\right)}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}t^{\alpha-1}s^{\beta-1}f\left(tb+(1-t)a, sd+(1-s)c\right)dsdt,
\end{aligned}$$

$$\begin{aligned}
I_9 &= -\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}}t^{\alpha}s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sd+(1-s)c\right)dsdt \qquad (12)\\
&= \frac{1}{2^{\alpha+\beta}\left(b-a\right)\left(d-c\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&- \frac{\beta}{2^{\alpha}\left(b-a\right)\left(d-c\right)}\int_{0}^{\frac{1}{2}}s^{\beta-1}f\left(\frac{a+b}{2}, sd+(1-s)c\right)ds\\
&-\frac{\alpha}{2^{\beta}\left(b-a\right)\left(d-c\right)}\int_{0}^{\frac{1}{2}}t^{\alpha-1}f\left(ta+(1-t)b, \frac{c+d}{2}\right)dt\\
&+\frac{\alpha\beta}{\left(b-a\right)\left(d-c\right)}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}}t^{\alpha-1}s^{\beta-1}f\left(ta+(1-t)b, sd+(1-s)c\right)dsdt,
\end{aligned}$$

$$\begin{aligned}
I_{10} &= -\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1}t^{\alpha}\left(s^{\beta}-1\right)\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sd+(1-s)c\right)dsdt \qquad (13)\\
&= \frac{\left(2^{\beta}-1\right)}{2^{\alpha+\beta}\left(b-a\right)\left(d-c\right)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{\beta}{2^{\alpha}\left(b-a\right)\left(d-c\right)}\int_{\frac{1}{2}}^{1}s^{\beta-1}f\left(\frac{a+b}{2}, sd+(1-s)c\right)ds\\
&-\frac{\alpha\left(2^{\beta}-1\right)}{2^{\beta}\left(b-a\right)\left(d-c\right)}\int_{0}^{\frac{1}{2}}t^{\alpha-1}f\left(ta+(1-t)b, \frac{c+d}{2}\right)dt
\end{aligned}$$

$$+\frac{\alpha\beta}{(b-a)(d-c)}\int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sd+(1-s)c)\,dsdt,$$

$$\begin{aligned}I_{11} &= \int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}(1-t^\alpha)s^\beta\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sd+(1-s)c)\,dsdt \qquad (14)\\
&= \frac{(2^\alpha-1)}{2^{\alpha+\beta}(b-a)(d-c)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{\beta(2^\alpha-1)}{2^\alpha(b-a)(d-c)}\int_0^{\frac{1}{2}}s^{\beta-1}f\left(\frac{a+b}{2},sc+(1-s)d\right)ds\\
&-\frac{\alpha}{2^\beta(b-a)(d-c)}\int_{\frac{1}{2}}^1 t^{\alpha-1}f\left(ta+(1-t)b,\frac{c+d}{2}\right)dt\\
&+\frac{\alpha\beta}{(b-a)(d-c)}\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sd+(1-s)c)\,dsdt,\end{aligned}$$

$$\begin{aligned}I_{12} &= \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1(1-t^\alpha)\left(s^\beta-1\right)\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sd+(1-s)c)\,dsdt \qquad (15)\\
&= \frac{(2^\alpha-1)\left(2^\beta-1\right)}{2^{\alpha+\beta}(b-a)(d-c)}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&-\frac{\beta(2^\alpha-1)}{2^\alpha(b-a)(d-c)}\int_{\frac{1}{2}}^1 s^{\beta-1}f\left(\frac{a+b}{2},sd+(1-s)c\right)ds\\
&-\frac{\alpha\left(2^\beta-1\right)}{2^\beta(b-a)(d-c)}\int_{\frac{1}{2}}^1 t^{\alpha-1}f\left(ta+(1-t)b,\frac{c+d}{2}\right)dt\\
&+\frac{\alpha\beta}{(b-a)(d-c)}\int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1 t^{\alpha-1}s^{\beta-1}f(ta+(1-t)b,sd+(1-s)c)\,dsdt,\end{aligned}$$

$$\begin{aligned}I_{13} &= -\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}t^\alpha s^\beta\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a,sc+(1-s)d)\,dsdt \qquad (16)\\
&= frac12^{\alpha+\beta}(b-a)(d-c)f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\end{aligned}$$

$$-\frac{\beta}{2^{\alpha}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} s^{\beta-1} f\left(\frac{a+b}{2}, sc+(1-s)d\right) ds$$

$$-\frac{\alpha}{2^{\beta}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} t^{\alpha-1} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt$$

$$+\frac{\alpha\beta}{(b-a)(d-c)}\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} t^{\alpha-1}s^{\beta-1} f\left(tb+(1-t)a, sc+(1-s)d\right) dsdt,$$

$$I_{14} = \int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sc+(1-s)d\right) dsdt \tag{17}$$

$$= \frac{\left(2^{\beta}-1\right)}{2^{\alpha+\beta}(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$-\frac{\beta}{2^{\alpha}(b-a)(d-c)}\int_{\frac{1}{2}}^{1} s^{\beta-1} f\left(\frac{a+b}{2}, sc+(1-s)d\right) ds$$

$$-\frac{\alpha\left(2^{\beta}-1\right)}{2^{\beta}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} t^{\alpha-1} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt$$

$$+\frac{\alpha\beta}{(b-a)(d-c)}\int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha-1}s^{\beta-1} f\left(tb+(1-t)a, sc+(1-s)d\right) dsdt,$$

$$I_{15} = -\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} (t^{\alpha}-1)s^{\beta}\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sc+(1-s)d\right) dsdt \tag{18}$$

$$= \frac{(2^{\alpha}-1)}{2^{\alpha+\beta}(b-a)(d-c)} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$- \frac{\beta(2^{\alpha}-1)}{2^{\alpha}(b-a)(d-c)}\int_{0}^{\frac{1}{2}} s^{\beta-1} f\left(\frac{a+b}{2}, sc+(1-s)d\right) ds$$

$$- \frac{\alpha}{2^{\beta}(b-a)(d-c)}\int_{\frac{1}{2}}^{1} t^{\alpha-1} f\left(tb+(1-t)a, \frac{c+d}{2}\right) dt$$

$$+ \frac{\alpha\beta}{(b-a)(d-c)}\int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}} t^{\alpha-1}s^{\beta-1} f\left(tb+(1-t)a, sc+(1-s)d\right) dsdt,$$

and

$$\begin{aligned} I_{16} &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (t^{\alpha}-1)\left(1-s^{\beta}\right)\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\,dsdt \qquad (19)\\ &= \frac{(2^{\alpha}-1)\left(2^{\beta}-1\right)}{2^{\alpha+\beta}(b-a)(d-c)} f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\ &- \frac{\beta\,(2^{\alpha}-1)}{2^{\alpha}(b-a)(d-c)}\int_{\frac{1}{2}}^{1} s^{\beta-1} f\left(\frac{a+b}{2}, sc+(1-s)d\right)ds\\ &- \frac{\alpha\left(2^{\beta}-1\right)}{2^{\beta}(b-a)(d-c)}\int_{\frac{1}{2}}^{1} t^{\alpha-1} f\left(tb+(1-t)a, \frac{c+d}{2}\right)dt\\ &+ \frac{\alpha\beta}{(b-a)(d-c)}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} t^{\alpha-1}s^{\beta-1} f(tb+(1-t)a, sc+(1-s)d)\,dsdt. \end{aligned}$$

From (4)-(19), using the change of the variable $x = ta + (1-t)b$ and $y = sc + (1-s)d$ for $t, s \in [0,1]$, we can write

$$I_1 + I_2 + I_3 + ... + I_{16} \qquad (20)$$

$$\begin{aligned} &= \frac{4}{(b-a)(d-c)} f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{2\Gamma(\alpha+1)}{(b-a)^{\alpha+1}(d-c)}\left[J_{a+}^{\alpha} f\left(b,\frac{c+d}{2}\right) + J_{b-}^{\alpha} f\left(a,\frac{c+d}{2}\right)\right]\\ &- \frac{2\Gamma(\beta+1)}{(b-a)(d-c)^{\beta+1}}\left[J_{c+}^{\beta} f\left(\frac{a+b}{2},d\right) + J_{d-}^{\beta} f\left(\frac{a+b}{2},c\right)\right]\\ &+ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{(b-a)^{\alpha+1}(d-c)^{\beta+1}}\left[J_{a+,c+}^{\alpha,\beta} f(b,d) + J_{a+,d-}^{\alpha,\beta} f(b,c) + J_{b-,c+}^{\alpha,\beta} f(a,d) + J_{b-,d-}^{\alpha,\beta} f(a,c)\right]. \end{aligned}$$

Multiplying both sides of (20) by $\frac{(b-a)(d-c)}{4}$, we obtain (3), which completes the proof. □

Next, we start to state the first theorem containing the Hermite-Hadamard type inequality for fractional integrals.

Theorem 2.1. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with with $0 \le a < b,\ 0 \le c < d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned} &\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \left[\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f\left(b,\frac{c+d}{2}\right) + J_{b-}^{\alpha} f\left(a,\frac{c+d}{2}\right)\right]\right.\right.\\ &\left.+ \frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left[J_{c+}^{\beta} f\left(\frac{a+b}{2},d\right) + J_{d-}^{\beta} f\left(\frac{a+b}{2},c\right)\right]\right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\ &\left.\times\left[J_{a+,c+}^{\alpha,\beta} f(b,d) + J_{a+,d-}^{\alpha,\beta} f(b,c) + J_{b-,c+}^{\alpha,\beta} f(a,d) + J_{b-,d-}^{\alpha,\beta} f(a,c)\right]\right|\\ &\le \frac{(b-a)(d-c)}{4}\left(\frac{1}{2} + \frac{1-2^{\alpha}}{2^{\alpha}(\alpha+1)}\right)\left(\frac{1}{2} + \frac{1-2^{\beta}}{2^{\beta}(\beta+1)}\right) \end{aligned}$$

$$\times\left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right).$$

Proof. From Lemma 2.1, we have

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\left[\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f\left(b,\frac{c+d}{2}\right)+J^{\alpha}_{b-}f\left(a,\frac{c+d}{2}\right)\right]\right.\right. \tag{21}$$

$$+\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left[J^{\beta}_{c+}f\left(\frac{a+b}{2},d\right)+J^{\beta}_{d-}f\left(\frac{a+b}{2},c\right)\right]\Bigg]+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}$$

$$\times\left.\left[J^{\alpha,\beta}_{a+,c+}f(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)\right]\right|$$

$$\leq\frac{(b-a)(d-c)}{4}\{L_1+L_2...+L_{16}\}.$$

Since $\left|\frac{\partial^2 f}{\partial t\partial s}\right|$ is convex function on the co-ordinates on Δ, by calculating the integrals in above inequality, then one has:

$$\begin{aligned}
L_1 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b,sc+(1-s)d)\right|dsdt \qquad (22)\\
&\leq \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\right.\\
&+ \left.t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+(1-s)(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right\}dsdt\\
&= \frac{1}{2^{\alpha+\beta+4}(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\
&+ \frac{\alpha+3}{2^{\alpha+\beta+4}(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\\
&+ \frac{\beta+3}{2^{\alpha+\beta+4}(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&+ \frac{(\alpha+3)(\beta+3)}{2^{\alpha+\beta+4}(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned}$$

$$
\begin{aligned}
L_2 &= \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)\right| dsdt \qquad (23) \\
&\leq \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right| \\
&+ \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right| \\
&+ \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right| \\
&+ \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_3 &= \int_{\frac{1}{2}}^{1} \int_0^{\frac{1}{2}} (1-t^{\alpha})s^{\beta}\left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)\right| dsdt \qquad (24) \\
&\leq \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right| \\
&+ \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right| \\
&+ \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right| \\
&+ \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_4 &= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sc+(1-s)d)\right| dsdt \qquad (25) \\
&\leq \left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right| \\
&+ \left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right| \\
&+ \left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right| \\
&+ \left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_5 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^\alpha s^\beta \left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sd+(1-s)c\right)\right| dsdt \\
&\leq \frac{(\alpha+3)(\beta+3)}{2^{\alpha+\beta+4}(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \\
&+ \frac{\beta+3}{2^{\alpha+\beta+4}(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
&+ \frac{\alpha+3}{2^{\alpha+\beta+4}(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
&+ \frac{1}{2^{\alpha+\beta+4}(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned} \tag{26}
$$

$$
L_6 = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^\alpha\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sd+(1-s)c\right)\right| dsdt \tag{27}
$$

$$
\begin{aligned}
&\leq \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \\
&+ \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
&+ \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
&+ \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_7 &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^\alpha)s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sd+(1-s)c\right)\right| dsdt \\
&\leq \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right| \\
&+ \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right| \\
&+ \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right| \\
&+ \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned} \tag{28}
$$

$$
\begin{aligned}
L_8 &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sd+(1-s)c)\right| dsdt \qquad (29)\\
&\leq \left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\
&+ \left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\\
&+ \left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&+ \left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_9 &= \int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}} t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sd+(1-s)c)\right| dsdt \qquad (30)\\
&\leq \frac{\beta+3}{2^{\alpha+\beta+4}(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\
&+ \frac{(\alpha+3)(\beta+3)}{2^{\alpha+\beta+4}(\alpha+1)(\alpha+2)(\beta+1)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\\
&+ \frac{1}{2^{\alpha+\beta+4}(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&+ \frac{\alpha+3}{2^{\alpha+\beta+4}(\alpha+1)(\alpha+2)(\beta+2)}\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_{10} &= \int_{0}^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sd+(1-s)c)\right| dsdt \qquad (31)\\
&\leq \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\
&+ \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\\
&+ \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&+ \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_{11} &= \int_{\frac{1}{2}}^{1}\int_{0}^{\frac{1}{2}}(1-t^{\alpha})s^{\beta}\left|\frac{\partial^{2}f}{\partial t\partial s}\left(ta+(1-t)b,sd+(1-s)c\right)\right|dsdt \qquad (32)\\
&\leq \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(a,c\right)\right|\\
&+ \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(b,c\right)\right|\\
&+ \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(a,d\right)\right|\\
&+ \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(b,d\right)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_{12} &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(ta+(1-t)b,sd+(1-s)c\right)\right|dsdt \qquad (33)\\
&\leq \left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(a,c\right)\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ \left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(b,c\right)\right|\\
&+ \left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(a,d\right)\right|\\
&+ \left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^{2}f}{\partial t\partial s}\left(b,d\right)\right|,
\end{aligned}
$$

$$
\begin{aligned}
L_{13} &= \int_{0}^{\frac{1}{2}}\int_{0}^{\frac{1}{2}}t^{\alpha}s^{\beta}\left|\frac{\partial^{2}f}{\partial t\partial s}\left(tb+(1-t)a,sc+(1-s)d\right)\right|dsdt \qquad (34)\\
&\leq \frac{\alpha+3}{2^{\alpha+\beta+4}\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^{2}f}{\partial t\partial s}\left(a,c\right)\right|\\
&+ \frac{1}{2^{\alpha+\beta+4}\left(\alpha+2\right)\left(\beta+2\right)}\left|\frac{\partial^{2}f}{\partial t\partial s}\left(b,c\right)\right|\\
&+ \frac{\left(\alpha+3\right)\left(\beta+3\right)}{2^{\alpha+\beta+4}\left(\alpha+1\right)\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^{2}f}{\partial t\partial s}\left(a,d\right)\right|\\
&+ \frac{\beta+3}{2^{\alpha+\beta+4}\left(\alpha+2\right)\left(\beta+1\right)\left(\beta+2\right)}\left|\frac{\partial^{2}f}{\partial t\partial s}\left(b,d\right)\right|,
\end{aligned}
$$

$$\begin{aligned}
L_{14} &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^\alpha\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a,sc+(1-s)d)\right|dsdt \quad (35)\\
&\leq \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\
&+ \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\\
&+ \frac{(\alpha+3)}{2^{\alpha+2}(\alpha+1)(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&+ \frac{1}{2^{\alpha+2}(\alpha+2)}\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned}$$

$$\begin{aligned}
L_{15} &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}}(1-t^\alpha)s^\beta\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a,sc+(1-s)d)\right|dsdt \quad (36)\\
&\leq \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\
&+ \frac{1}{2^{\beta+2}(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\\
&+ \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&+ \frac{(\beta+3)}{2^{\beta+2}(\beta+1)(\beta+2)}\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|,
\end{aligned}$$

and

$$\begin{aligned}
L_{16} &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}(1-t^\alpha)\left(1-s^\beta\right)\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a,sc+(1-s)d)\right|dsdt \quad (37)\\
&\leq \left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|\\
&+\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{3}{8}+\frac{1-2^{\beta+2}}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|\\
&+\left(\frac{1}{8}+\frac{1-2^{\alpha+1}}{2^{\alpha+1}(\alpha+1)}+\frac{2^{\alpha+2}-1}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|\\
&+\left(\frac{3}{8}+\frac{1-2^{\alpha+2}}{2^{\alpha+2}(\alpha+2)}\right)\left(\frac{1}{8}+\frac{1-2^{\beta+1}}{2^{\beta+1}(\beta+1)}+\frac{2^{\beta+2}-1}{2^{\beta+2}(\beta+2)}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|.
\end{aligned}$$

From (22)-(37), we have

$$L_1+L_2...+L_{16} = \left(\frac{1}{2}+\frac{1-2^{\alpha}}{2^{\alpha}(\alpha+1)}\right)\left(\frac{1}{2}+\frac{1-2^{\beta}}{2^{\beta}(\beta+1)}\right) \times \left(\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|\right). \tag{38}$$

Substituting (38) in (21), we obtain desired result. □

Theorem 2.2. *Let $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be a partial differentiable mapping on $\Delta:=[a,b]\times[c,d]$ in $\mathbb{R}^2$ with $0\le a<b,\ 0\le c<d$. If $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$, $q>1$, is a convex function on the co-ordinates on Δ, then one has the inequalities:*

$$\begin{aligned}
&\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right)-\left[\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f\left(b,\frac{c+d}{2}\right)+J^{\alpha}_{b-}f\left(a,\frac{c+d}{2}\right)\right]\right.\right.\\
&+\frac{\Gamma(\beta+1)}{2\,(d-c)^{\beta}}\left[J^{\beta}_{c+}f\left(\frac{a+b}{2},d\right)+J^{\beta}_{d-}f\left(\frac{a+b}{2},c\right)\right]\right]+\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4\,(b-a)^{\alpha}\,(d-c)^{\beta}}\\
&\times\left.\left[J^{\alpha,\beta}_{a+,c+}f(b,d)+J^{\alpha,\beta}_{a+,d-}f(b,c)+J^{\alpha,\beta}_{b-,c+}f(a,d)+J^{\alpha,\beta}_{b-,d-}f(a,c)\right]\right|^{\frac{1}{q}}\\
&\le \frac{(b-a)\,(d-c)}{4}A\\
&\times\left\{\left(\frac{9\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}\\
&+\left(\frac{3\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}\\
&+\left.\left(\frac{3\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$A = \left[\left(\frac{1}{2}+\frac{1-2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)}\right)^{\frac{1}{p}}+\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)^{\frac{1}{p}}\right] \times \left[\left(\frac{1}{2}+\frac{1-2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)}\right)^{\frac{1}{p}}+\left(\frac{1}{2^{p\beta+1}(p\beta+1)}\right)^{\frac{1}{p}}\right]$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1, we have

$$
\begin{aligned}
&\left| f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \left[\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f\left(b,\frac{c+d}{2}\right) + J_{b-}^{\alpha} f\left(a,\frac{c+d}{2}\right)\right]\right.\right. \qquad (39)\\
&+ \left.\frac{\Gamma(\beta+1)}{2(d-c)^{\beta}}\left[J_{c+}^{\beta} f\left(\frac{a+b}{2},d\right) + J_{d-}^{\beta} f\left(\frac{a+b}{2},c\right)\right]\right] + \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{4(b-a)^{\alpha}(d-c)^{\beta}}\\
&\times \left.\left[J_{a+,c+}^{\alpha,\beta} f(b,d) + J_{a+,d-}^{\alpha,\beta} f(b,c) + J_{b-,c+}^{\alpha,\beta} f(a,d) + J_{b-,d-}^{\alpha,\beta} f(a,c)\right]\right|\\
&\le \frac{(b-a)(d-c)}{4}\left\{L_1 + L_2 ... + L_{16}\right\}.
\end{aligned}
$$

By using the well known Hölder's inequality for double integrals and $\left|\frac{\partial^2 f}{\partial t\partial s}\right|^q$ is convex function on the co-ordinates on Δ, we get

$$
\begin{aligned}
L_1 &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right| dsdt \qquad (40)\\
&\le \left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{p\alpha}s^{p\beta}dsdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}\\
&\le \left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{p\alpha}s^{p\beta}dsdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left\{ts\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + s(1-t)\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q\right.\right.\\
&+ \left.\left. t(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + (1-t)(1-s)\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q\right\}\right)^{\frac{1}{q}}\\
&\le \left(\frac{1}{2^{p\alpha+p\beta+2}(p\alpha+1)(p\beta+1)}\right)^{\frac{1}{p}}\\
&\times \left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q + 3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q + 3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q + 9\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}},
\end{aligned}
$$

$$
L_2 = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right| dsdt \qquad (41)
$$

$$
\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{p\alpha}\left(1-s^{\beta}\right)^p dsdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\frac{\partial^2 f}{\partial t\partial s}(ta+(1-t)b, sc+(1-s)d)\right|^q dsdt\right)^{\frac{1}{q}}
$$

$$\leq \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^{p\alpha} \left(1 - s^{p\beta}\right) dsdt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left\{ ts \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + s(1-t) \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q \right.\right.$$

$$\left.\left. + \ t(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + (1-t)(1-s) \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q \right\} \right)^{\frac{1}{q}}$$

$$\leq \left[\left(\frac{1}{2} + \frac{1 - 2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)} \right) \left(\frac{1}{2^{p\alpha+1}(p\alpha+1)} \right) \right]^{\frac{1}{p}}$$

$$\times \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right)^{\frac{1}{q}},$$

$$\begin{aligned} L_3 &= \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} (1-t^\alpha) s^\beta \left| \frac{\partial^2 f}{\partial t \partial s} \left(ta + (1-t)b, sc + (1-s)d \right) \right| dsdt \\ &\leq \left[\left(\frac{1}{2} + \frac{1 - 2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)} \right) \left(\frac{1}{2^{p\beta+1}(p\beta+1)} \right) \right]^{\frac{1}{p}} \\ &\times \left(\frac{3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + 9 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right)^{\frac{1}{q}}, \end{aligned} \tag{42}$$

$$\begin{aligned} L_4 &= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 (1-t^\alpha) \left(1 - s^\beta\right) \left| \frac{\partial^2 f}{\partial t \partial s} \left(ta + (1-t)b, sc + (1-s)d \right) \right| dsdt \\ &\leq \left[\left(\frac{1}{2} + \frac{1 - 2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)} \right) \left(\frac{1}{2} + \frac{1 - 2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)} \right) \right]^{\frac{1}{p}} \\ &\times \left(\frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right)^{\frac{1}{q}}, \end{aligned} \tag{43}$$

$$\begin{aligned} L_5 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^\alpha s^\beta \left| \frac{\partial^2 f}{\partial t \partial s} \left(tb + (1-t)a, sd + (1-s)c \right) \right| dsdt \\ &\leq \left(\frac{1}{2^{p\alpha+p\beta+2}(p\alpha+1)(p\beta+1)} \right)^{\frac{1}{p}} \\ &\times \left(\frac{9 \left| \frac{\partial^2 f}{\partial t \partial s}(a,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(b,c) \right|^q + 3 \left| \frac{\partial^2 f}{\partial t \partial s}(a,d) \right|^q + \left| \frac{\partial^2 f}{\partial t \partial s}(b,d) \right|^q}{64} \right)^{\frac{1}{q}}, \end{aligned} \tag{44}$$

$$
\begin{aligned}
L_6 &= \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(tb+(1-t)a, sd+(1-s)c)\right| dsdt \qquad (45) \\
&\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)}\right)\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)\right]^{\frac{1}{p}} \\
&\times \left(\frac{3\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
L_7 &= \int_{\frac{1}{2}}^{1} \int_0^{\frac{1}{2}} (1-t^{\alpha})s^{\beta}\left|\frac{\partial^2 f}{\partial t \partial s}(tb+(1-t)a, sd+(1-s)c)\right| dsdt \qquad (46) \\
&\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)}\right)\left(\frac{1}{2^{p\beta+1}(p\beta+1)}\right)\right]^{\frac{1}{p}} \\
&\times \left(\frac{3\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
L_8 &= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{1} (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t \partial s}(tb+(1-t)a, sd+(1-s)c)\right| dsdt \qquad (47) \\
&\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)}\right)\left(\frac{1}{2}+\frac{1-2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)}\right)\right]^{\frac{1}{p}} \\
&\times \left(\frac{\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
L_9 &= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t \partial s}(ta+(1-t)b, sd+(1-s)c)\right| dsdt \qquad (48) \\
&\leq \left(\frac{1}{2^{p\alpha+p\beta+2}(p\alpha+1)(p\beta+1)}\right)^{\frac{1}{p}} \\
&\times \left(\frac{3\left|\frac{\partial^2 f}{\partial t \partial s}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t \partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t \partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t \partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}},
\end{aligned}
$$

$$\begin{aligned} L_{10} &= \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sd+(1-s)c\right)\right| dsdt \\ &\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)}\right)\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)\right]^{\frac{1}{p}} \\ &\times \left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}, \end{aligned} \tag{49}$$

$$\begin{aligned} L_{11} &= \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^{\alpha})s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sd+(1-s)c\right)\right| dsdt \\ &\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)}\right)\left(\frac{1}{2^{p\beta+1}(p\beta+1)}\right)\right]^{\frac{1}{p}} \\ &\times \left(\frac{9\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}, \end{aligned} \tag{50}$$

$$\begin{aligned} L_{12} &= \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}\left(ta+(1-t)b, sd+(1-s)c\right)\right| dsdt \\ &\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)}\right)\left(\frac{1}{2}+\frac{1-2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)}\right)\right]^{\frac{1}{p}} \\ &\times \left(\frac{3\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}, \end{aligned} \tag{51}$$

$$\begin{aligned} L_{13} &= \int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}} t^{\alpha}s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}\left(tb+(1-t)a, sc+(1-s)d\right)\right| dsdt \\ &\leq \left(\frac{1}{2^{p\alpha+p\beta+2}(p\alpha+1)(p\beta+1)}\right)^{\frac{1}{p}} \\ &\times \left(\frac{3\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}, \end{aligned} \tag{52}$$

$$L_{14} = \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^{1} t^{\alpha}\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\right| dsdt \tag{53}$$

$$\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)}\right)\left(\frac{1}{2^{p\alpha+1}(p\alpha+1)}\right)\right]^{\frac{1}{p}}$$

$$\times \left(\frac{9\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}},$$

$$L_{15} = \int_{\frac{1}{2}}^{1}\int_0^{\frac{1}{2}} (1-t^{\alpha})s^{\beta}\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\right| dsdt \tag{54}$$

$$\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)}\right)\left(\frac{1}{2^{p\beta+1}(p\beta+1)}\right)\right]^{\frac{1}{p}}$$

$$\times \left(\frac{\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}},$$

and

$$L_{16} = \int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1} (1-t^{\alpha})\left(1-s^{\beta}\right)\left|\frac{\partial^2 f}{\partial t\partial s}(tb+(1-t)a, sc+(1-s)d)\right| dsdt \tag{55}$$

$$\leq \left[\left(\frac{1}{2}+\frac{1-2^{p\alpha+1}}{2^{p\alpha+1}(p\alpha+1)}\right)\left(\frac{1}{2}+\frac{1-2^{p\beta+1}}{2^{p\beta+1}(p\beta+1)}\right)\right]^{\frac{1}{p}}$$

$$\times \left(\frac{3\left|\frac{\partial^2 f}{\partial t\partial s}(a,c)\right|^q+9\left|\frac{\partial^2 f}{\partial t\partial s}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial t\partial s}(a,d)\right|^q+3\left|\frac{\partial^2 f}{\partial t\partial s}(b,d)\right|^q}{64}\right)^{\frac{1}{q}}.$$

Here, we use $(A-B)^p \leq A^p - B^p$, for any $A > B \geq 0$ and $q \geq 1$.

Writing (40)-(55) in (39), we obtain desired result. □

3. Conclusions

In this study, we obtained some Hermite-Hadamard type inequalities via Riemann-Liouville fractional integrals for two variables using co-ordinated convex functions. The left-hand side of the fractional Hermite-Hadamard type inequality on the coordinates is derived. These results can be viewed as a refinement of the previously known results.

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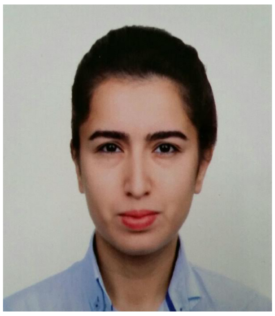
Tuba TUNÇ - graduated from Karadeniz Technical University, Trabzon, Turkey in 2011. She received her M.Sc. from Karadeniz Tecnical University in 2013. Since 2014 she is a Ph.D. student and works as a Research Assistant at Duzce University. Her research interests area is local fractional integral.

Mehmet Zeki Ssrikaya - received his B.Sc. (Maths), M.Sc. (Maths) and Ph.D. (Maths) degrees from Afyon Kocatepe University, Afyonkarahisar, Turkey in 2000, 2002 and 2007 respectively. At present, he is working as a Professor in the Department of Mathematics at Duzce University (Turkey) and as a Head of Department. His research interests are the theory of inequalities, potential theory, integral equations and transforms, special functions, time-scales.



Hatice Yaldiz - received her Ph.D. degree in Applied Mathematics in 2016 from Duzce University, Duzce, Turkey. She has been in Western Kentucky University as a visiting researcher in 2015. Her major research interests include inequalities, fractional theory, discrete analysis, time-scales.